Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Pure and Applied Algebra 185 (2003) 1–5

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Real varieties associated to complex varieties

E. Ballico*

Department of Mathematics, University of Trento, 38050 Povo (TN), Italy

Received 14 October 2001; received in revised form 14 March 2003; accepted 8 April 2003

Communicated by M.-F. Roy

Abstract

For any irreducible and reduced compact complex space or any integral projective variety, let X^σ be its complex conjugate. Let $S(X)$ be the set of all W such that $W \times W^\sigma \cong X \times X^\sigma$. We say that $S(X)$ is trivial if for every $W \in S(X)$ there are Y, Z such that $X \cong Y \times Z$ and $W \cong Y \times Z^\sigma$. A torsion bundle over a torus is the total space of a fibre bundle with finite structure group over a positive dimensional torus. Here we prove that if X has no torsion bundle over a torus as a factor, then $S(X)$ is trivial.

MSC: Primary: 14P99; secondary: 32J18

1. Introduction

Let $\{1, \sigma\}$ be the Galois group of the field extension $\mathbf{C} \setminus \mathbf{R}$. In general we will use σ to denote its action on algebraic varieties defined over Spec($\mathbf{R}$) and their related objects and often complex conjugation on reduced complex spaces [9, p. 2]. For any reduced complex compact space let X^σ be its complex conjugate in the sense of [9, Section 2]. For any integral algebraic variety X over Spec($\mathbf{C}$), let X^σ be its complex conjugate in the sense of Weil ([10, Section 1.3], or [6]). If X is projective, say $X \subseteq \mathbf{P}^{\mathrm{N}}(\mathbf{C})$, one can define $X \subseteq \mathbf{P}^{\mathrm{N}}(\mathbf{C})$ taking as defining equations the complex conjugations of the homogeneous equations defining X (see [1, Section 2], or [6, Section 2]).

* Fax: +39-0461-881-624.

E-mail address: ballico@science.unitn.it (E. Ballico).

0022-4049/03/$ - see front matter © 2003 Elsevier B.V. All rights reserved.

doi:10.1016/S0022-4049(03)00108-7

Set ${}_{\mathbf{R}}X := X \times X^{\sigma}$. The quotient of the variety ${}_{\mathbf{R}}X$ by the action of $\{1, \sigma\}$ defined by $\sigma((a,b)) = (\sigma(b), \sigma(a))$ is the variety obtained from X by the restriction of scalars $\mathbf{C} \setminus \mathbf{R}$ in the sense of Weil (see [10, Section 1.3], or [4, Exp. 195]).

Let X be a compact reduced and irreducible compact space. Following [5] we will say that X is a torsion bundle over a torus if it is the total space of a fibre bundle with finite structure group over a positive dimensional torus, i.e. if there is a positive dimensional torus T, a reduced and irreducible compact space X_0, a surjective holomorphic map $f : X \to T$, an étale covering $m : T' \to T$ with T' torus and m the quotient map by a finite subgroup $K \subset T$ and an action of K onto X_0 such that the fiber product of m and f identifies X with $(X_0 \times T')/K$. If X is projective, we may use the same definition in the category of projective varieties because if X is projective, then T, T' and X_0 are projective.

Definition 1.1. For any integral complex projective variety X let $S(X)$ be the set of the isomorphism classes of all complex projective varieties W such that ${}_{\mathbf{R}}X \cong_{\mathbf{R}} W$. We will say that $S(X)$ is trivial if for every $W \in S(X)$ there are Y, Z such that $X \cong Y \times Z$ and $W \cong Y \times Z^{\sigma}$.

We can make the same definition in the category of reduced and irreducible compact complex spaces. If X is a torsion bundle over a torus, then X^{σ} is a torsion bundle over a torus and hence ${}_{\mathbf{R}}X$ is a torsion bundle over a torus.

Remark 1.2. Let X, Y be compact reduced and irreducible complex spaces. If $X \times Y$ is projective (resp. algebraic, resp. Moishezon), then X and Y are projective (resp. algebraic, resp. Moishezon).

Remark 1.3. Let X be a compact reduced and irreducible complex space. By [5, Theorems 1.1 and 1.2], there are compact complex spaces Y, Z (uniquely determined up to biholomorphisms) such that Y is a torsion bundle over a tours, Z has no torsion bundle over a torus as a factor and X is biholomorphic to $Y \times Z$. We will call Y the toral part $\mathrm{Tor}(Y)$ of X. By Remark 1.2 if X is projective (resp. algebraic, resp. Moishezon), then Y and Z are projective (resp. algebraic, resp. Moishezon).

In this paper we will prove the following result.

Theorem 1.4. *Let X be either an integral projective variety or a reduced and irreducible compact complex space. Assume that the toral part of X is trivial. Then $S(X)$ is trivial.*

To prove Theorem 1.4 we will use some results on the Cancellation Problem for the category of reduced and irreducible compact complex space and the category of integral projective varieties (see [2,3,5,7,8] and references therein). We will use this opportunity to point out the extension to the latter category of some results proved in the former category (see Section 3).

2. Proof of Theorem 1.4

We collect here two trivial remarks which show how to construct varieties with $S(X)$ trivial but $S(X) \neq \{X, X^\sigma\}$.

Remark 2.1. We have $(X^\sigma)^\sigma \cong X$ (as varieties over Spec($\mathbf{C}$)) and $A \times B \cong B \times A$. Hence ${}_{\mathbf{R}}X \cong {}_{\mathbf{R}}(X^\sigma) \cong ({}_{\mathbf{R}}X)^\sigma$.

The following observation was our motivation for Definition 2.1.

Remark 2.2. Let A, B be complex algebraic varieties or compact reduced and irreducible complex spaces. Set $X_1 := A \times (B^\sigma)$ and $X_2 := (A^\sigma) \times B$. By Remark 2.1 we have ${}_{\mathbf{R}}X_1 \cong A \times (B^\sigma) \times (A^\sigma) \times B \cong {}_{\mathbf{R}}X_2$.

Proof of Theorem 1.4. Fix $W \in S(X)$. Since $(Y \times Z)^\sigma \cong Y^\sigma \times Z^\sigma$, the toral part of $X \times X^\sigma$ is trivial. Hence the toral parts of W and W^σ are trivial. Hence the triviality of $S(X)$ follows from the uniqueness part of the Cancellation Theorem [5, Theorem 1.2].

Remark 2.3. It is not true that if X is a torsion bundle over a tours, then $S(X)$ is not trivial. For instance, let E be a complex elliptic curve and $\rho({}_{\mathbf{R}}E)$ the number of isomorphism classes of complex structures of ${}_{\mathbf{R}}E$. We have $\rho({}_{\mathbf{R}}E) = 1$ (and hence $S(E)$ is trivial) if either E has complex multiplication and the class number of End(E) is 1 ([6, Corollary 1.4 and 1.5 and Remark 1.6]) or E has not complex multiplication and E^σ is not isogeneous to E ([6, Theorem 1.7]) or E has no complex multiplication and $j(E) \in \mathbf{R}$ ([6, Corollary 1.11]).

Definition 2.4. Let X, Y be either an integral projective variety or a reduced and irreducible compact complex space. We will say that $S(X \times Y)$ is trivial modulo $S(Y)$ if for every $W \in S(X \times Y)$ there are X_1, X_2, X_3, W_3 with $X \cong X_1 \times X_2 \times X_3$, $W \cong X_1 \times X_2^\sigma \times W_3$ and $W_3 \in S(Y)$.

3. Cancellation problem

Example 3.1. Let T be an n-dimensional complex torus and $O \in T$ its origin. Fix an integer $g > n$ and a general smooth curve X of genus g. G. Parigi constructed two fiber bundles F_1, F_2 on X with fiber T, F_1 not biholomorphic to F_2 but with $F_1 \times T$ biholomorphic to $F_2 \times T$ [7]. Here we will check that if T is algebraic, then we may take F_1 and F_2 projective and hence by GAGA that $F_1 \times T$ and $F_2 \times T$ are isomorphic projective varieties, while F_1 and F_2 are not isomorphic. For every integer $q \geqslant 2$, set $T[q] := \{P \in T\colon qP = O\}$. Hence $\mathrm{card}(T[q]) = q^{2n}$. $T[q]$ acts as a group of translations on T and hence it may be seen as a subgroup of the automorphism group of T as complex manifold. We fix an integer $q > 2^{2n} + 2$. The fiber bundle F_i, $1 \leqslant i \leqslant 2$, constructed in [7] is associated to a certain $\alpha_i \in \mathrm{H}^1(X, T[q])$. Since for every positive integer k $T[q]$ sends $T[kq]$ into itself, we see that the sets of points of order dividing kq on each fiber of the fibration $u_i : F_i \to T$ are preserved by the gluing data of the cocycle associated to α_i. Hence for any $P \in X$, the set $u_i^{-1}(P)[kq] \cong T[kq]$ is well-defined. The union of all $T[kq], P \in X$, gives an étale covering of X (perhaps not connected). This show the existence of an étale covering $\beta_i : Y_i \to X$ such that the T-fiber bundle $v_i : G_i \to Y_i$ induced by F_i is trivial. Hence if T is an Abelian variety, then G_i is a projective scheme. Since the map $m_i : G_i \to F_i$ induced by β_i, is étale and surjective, we immediately get that F_i is a Moishezon manifold. Since $T[kq]$ is abelian, the covering is Galois. Hence the covering $m_i : G_i \to F_i$ is Galois, say with group G. Fix an ample $L \in \mathrm{Pic}(G_i)$ and set $M := \bigotimes_{h \in G} h^*(L)$. Thus M is a G-invariant ample line bundle on G_i. Since m_i is ètale, M descends to an ample line bundle on F_i. Thus F_i is a projective manifold.

Theorem 3.2. *Let X be an integral projective variety. Assume that X is a torsion bundle over a torus. Then there exist integral projective varieties Y, Z such that $X \times Y \cong X \times Z$ but Y and Z are not isomorphic. If X is smooth, then we may take Y and Z smooth.*

Proof. Since X is a torsion bundle over a torus, there is a positive dimensional torus T, a reduced and irreducible compact space X_0, a surjective holomorphic map $f : X \to T$, an étale covering $m : T' \to T$ with T' torus and m the quotient map by a finite subgroup K of T and an action of K onto X_0 such that the fiber product of m and f identifies X with $(X_0 \times T')/K$. Since X is projective, T is projective. Since X is projective, its étale covering $X_0 \times T'$ is projective. Hence X_0 and T' are projective. By GAGA the holomorphic maps f and m are regular maps in the category of schemes. By [8, Theorem 2.2], there exist irreducible and reduced compact complex space Y, Z with $X \times Y$ biholomorphic to $X \times Z$ but Y and Z not biholomorphic. By GAGA it is sufficient to prove the existence of such complex spaces Y, Z which are projective. We just checked that T, T' and X_0 are projective. By Example 3.1 there are projective varieties $F_1(T')$ and $F_2(T')$ such that $F_1(T')$ and $F_2(T')$ are not isomorphic but $F_1(T') \times T' \cong F_2(T') \times T'$. Then it is shown in [8, Section 2], that $X \times F_1(T') \times X$ and $X \times F_2(T')$ are biholomorphic. Hence by GAGA we may take $Y = F_1(T')$ and $Z = F_2(T')$. The last assertion follows from the construction just given.

Theorem 3.3. *Let X be a smooth compact complex surface which is not algebraic. X is not simplificable (i.e. there are reduced and irreducible complex compact spaces Y, Z with $X \times Y$ biholomorphic to $X \times Z$ but Y not biholomorphic to Z) if and only if it is a torus.*

Proof. By [8] no torus is simplificable. By [5, Theorem 1.2], X is a torsion bundle over a torus. In particular there is a torus T with $\dim(T) > 0$ and a surjective holomorphic map $f : X \to T$ with isomorphic fibers. If $\dim(T) = 2$, then f is unramified and hence X is a torus. Assume $\dim(T) = 1$. Hence X has algebraic dimension 1. This implies that a general fiber of f is an elliptic curve and that every connected subcurve of X is contained in a fiber of f. As explained in [8, Proposition 1.1 and Remark 1.2], X admits another fibration over a curve, contradiction.

Acknowledgements

This research was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] J. Bochnak, J. Huisman, When is a complex elliptic curve the product of two real algebraic curves? Math. Ann. 293 (1992) 469–474.
[2] J. Brun, Sur la simplification dans les isomorphismes analytiques, Ann. Scuola Ec. Norm. Sup. 9 (4) (1976) 533–538.
[3] J. Brun, Sur la simplification par les variétés homogènes, Math. Ann. 230 (1977) 175–182.
[4] A. Grothendieck, Fondements de la géométrie algébrique (extraits du Séminaire Bourbaki 1957–1962), Exp. 195.
[5] C. Horst, Decomposition of compact complex varieties and the cancellation problem, Math. Ann. 271 (1985) 467–477.
[6] J. Huisman, The underlying real algebraic structure of complex elliptic curves, Math. Ann. 294 (1992) 19–35.
[7] G. Parigi, Sur la simplification par les tores complexes, J. Reine Angew. Math. 322 (1981) 42–52.
[8] G. Parigi, Caracterisation des variétés compactes simplifiables et applications aux surfaces algébriques, Math. Z. 190 (1985) 371–378.
[9] R. Silhol, Real Algebraic Surfaces, in: Lecture Notes in Mathematics, Vol. 1392, Springer, Berlin, Heidelberg, New York, 1986.
[10] A. Weil, Adeles and algebraic groups, in: Progress in Mathematics, Vol. 23, Birkhäuser, Boston, Basel, Stuttgart, 1982.